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Symmetry algebra of the anisotropic harmonic oscillator with commensurate frequencies

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Abstract. The symmetry algebra of the m-dimensional quantum anisotropic harmonic oscillator Hamiltonian H with commensurate frequencies is shown to be $u(m)$. Each eigenspace of H carries a single irreducible representation of $u(m)$. However, distinct eigenspaces can yield equivalent $u(m)$ representations. The dynamical algebra is the non-compact symplectic algebra $\mathrm{sp}(m, \mathbb{R})$. For $m=3$, the anisotropic Hamiltonian is relevant to superdeformed nuclei.

1. Introduction

The three-dimensional anisotropic harmonic oscillator defines the intrinsic state of rotating deformed nuclei in the Nilsson model [1]. For the recently discovered superdeformed nuclear states [2-8], the ratio of the axis lengths of the nuclear prolate spheroid is approximately 2:1. Hence, the oscillator frequencies $\omega_1 : \omega_2 : \omega_3$ are in the ratio 2:2:1. In this paper, the symmetry algebra of the m-dimensional anisotropic harmonic oscillator Hamiltonian with commensurate frequencies is shown to be $u(m)$. However, in contrast to the isotropic oscillator, $u(m)$ irreducible representations occur multiply in the anisotropic case. For example, the 2:1 superdeformed prolate spheroid yields a doubling of the u(3) irreducible representations. The symmetry algebra for the two-dimensional case was determined by Jauch and Hill [9] in the classical picture, and by Demkov [10] and Louck, Moshinsky and Wolf [11] in the quantum picture.

2. Symmetry algebra

Let $a_i^\dagger$ and a_i denote the usual oscillator creation and annihilation bosons in the ith Cartesian direction,

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{1}$$

Set the number operator $\hat{n}_i \equiv a_i^\dagger a_i$. Consider the m-dimensional anisotropic oscillator Hamiltonian with commensurate frequencies,

$$H \equiv \sum_{i=1}^{m} \frac{1}{p_i} \hat{n}_i \tag{2}$$

where p_i are positive integers.

A typical symmetry operator for H annihilates p_j quanta in the jth direction and creates p_i quanta in the ith direction. However, although the operators $D_{ij} \equiv (a_i^\dagger)^{p_i}(a_j)^{p_j}$ commute with the anisotropic oscillator Hamiltonian, they do not close under commutation. Indeed, the commutator $[D_{ij}, D_{kl}]$ is another operator of degree $(p_i+p_j+p_k+p_l-2)$. Except for the isotropic special case ($p_i = p_j = 1$ and/or $p_k = p_l = 1$), successive commutators produce new polynomial symmetry operators of increasingly higher degree in the bosons. Therefore, the symmetry algebra generated by the D_{ij} is infinite-dimensional unless all $p_i = 1$, i.e. the Hamiltonian is isotropic and the symmetry algebra is $\mathrm{u}(m)$.

In order to achieve a finite-dimensional symmetry algebra, it is necessary to define new bosons appropriate to the anisotropic oscillator. Consider first the one-dimensional case with the usual oscillator bosons $a^\dagger$ and a. Define the operators

$$\begin{aligned}\alpha^\dagger &= \hat{n}^{-1/2}a^\dagger \\ \alpha &= (\hat{n}+1)^{-1/2}a\end{aligned} \tag{3}$$

with $\hat{n} = a^\dagger a$.

When acting upon the oscillator state $|n\rangle$

$$\begin{aligned}\alpha^\dagger|n\rangle &= |n+1\rangle \\ \alpha|n\rangle &= \begin{cases}|n-1\rangle & n>0 \\ 0 & n=0.\end{cases}\end{aligned} \tag{4}$$

For each positive integer p, define the number operator modulo p by

$$[\hat{n}/p]|n\rangle = [n/p]|n\rangle \tag{5}$$

where $[n/p]$ denotes the whole integer part of the ratio n/p. Define the p-step creation and annihilation operators .

$$\begin{aligned}A(p)^\dagger &\equiv [\hat{n}/p]^{1/2}(\alpha^\dagger)^p \\ A(p) &\equiv \alpha^p[\hat{n}/p]^{1/2}.\end{aligned} \tag{6}$$

These multiboson operators have been applied to squeezed photon states [12, 13]. Their action on the orthonormal oscillator basis is given by

$$\begin{aligned}A(p)^\dagger|n\rangle &= ([n/p]+1)^{1/2}|n+p\rangle \\ A(p)|n\rangle &= \begin{cases}[n/p]^{1/2}\ |n-p\rangle & n\geqslant p \\ 0 & n<p.\end{cases}\end{aligned} \tag{7}$$

Hence, $A(p)^\dagger$ creates p-quanta and $A(p)$ destroys p-quanta,

$$\begin{aligned}[\hat{n}, A(p)^\dagger] &= pA(p)^\dagger \\ [\hat{n}, A(p)] &= -pA(p).\end{aligned} \tag{8}$$

The modulo-p number operator is given by

$$[\hat{n}/p] = A(p)^\dagger A(p). \tag{9}$$

The crucial result for this paper is that the p-step creation and annihilation operators satisfy boson commutation relations

$$[A(p), A(p)^\dagger] = 1. \tag{10}$$

It is interesting to note that, if the p-step operators were to be defined with the simple ratio $\hat{n}/p$ instead of the whole number operator $[\hat{n}/p]$, then the redefined operators would satisfy the boson commutation relation (10) everywhere except when acting upon the first p vectors $|n\rangle$, $0 \leq n < p$.

Returning now to the m-dimensional problem, it is evident that the m^2 operators

$$C_{ij} \equiv A_i(p_i)^\dagger A_j(p_j) + \tfrac{1}{2}\delta_{ij} \tag{11}$$

commute with the anisotropic oscillator Hamiltonian, where $A_i(p_i)^\dagger$ and $A_i(p_i)$ denote the p_i-step creation and destruction bosons acting upon the ith Cartesian coordinates. Since the p_i-step operators are bosons, the symmetry operators C_{ij} span the unitary Lie algebra u(m),

$$[C_{ij}, C_{kl}] = \delta_{jk}C_{il} - \delta_{il}C_{kj}. \tag{12}$$

3. Highest weight vectors

A u(m) highest weight vector in $\mathscr{L}^2(\mathbb{R}^m)$ must simultaneously be an eigenvector of C_{ii}, $1 \leq i \leq m$, and be annihilated by the $m(m-1)/2$ raising operators C_{ij}, $1 \leq i < j \leq m$. Thus, a highest weight vector is given by placing all excess quanta in the first Cartesian direction,

$$|n, \{q\}\rangle \equiv |np_1 + q_1\rangle \otimes |q_2\rangle \otimes \ldots \otimes |q_n\rangle \tag{13}$$

for $n = 0, 1, 2, \ldots$ and $\{q\} = \{q_1, q_2, \ldots, q_m\}$ is a sequence of m integers with $0 \leq q_i < p_i$. Clearly, $C_{ij}|n, \{q\}\rangle = 0$ for $i < j$, since there are at most $(p_j - 1)$ quanta in each of the j directions, $1 < j \leq m$. Moreover, $|n, \{q\}\rangle$ is a simultaneous eigenvector of the Cartan operators

$$C_{ii}|n, \{q\}\rangle = n\delta_{i1}|n, \{q\}\rangle. \tag{14}$$

Therefore, the irreducible $u(m)$ representation generated from the highest weight vector $|n, \{q\}\rangle$ has the weight $(n, 0, \ldots, 0)$. Furthermore, each such irreducible u(m) representation occurs with multiplicity $\Pi_{i=1}^m p_i$.

Is u(m) the maximal symmetry algebra? If all the integers p_i are relatively prime, $(p_i, p_j) = 1$ for every pair $i \neq j$, then u(m) is maximal and each eigenspace of the anisotropic oscillator spans a single irreducible representation. To demonstrate this, suppose by contrast that an eigenspace of H is the direct sum of two or more irreducible representations of u(m). Each such irreducible subspace contains a highest weight vector, which must be of the form $|n, \{q\}\rangle$. If two such highest weight vectors, $|n, \{q\}\rangle$ and $|n', \{q'\}\rangle$ are from the same eigenspace of the anisotropic oscillator, then

$$n' + \sum_{j=1}^{m} q_j'/p_j = n + \sum_{j=1}^{m} q_j/p_j. \tag{15}$$

If both sides of this equation are multiplied by $\Pi_{k\neq i}\, p_k$, then

$$(n'-n)\prod_{k\neq i} p_k + \sum_{j\neq i}(q_j' - q_j)\prod_{k\neq i,j} p_k = -\frac{(q_i' - q_i)}{p_i}\sum_{k\neq i} p_k. \tag{16}$$

Since the p_k are relatively prime, p_i does not divide $\Pi_{k\neq i}\, p_k$. Hence, although the left-hand side of (16) is an integer, the right-hand side is not integral unless $q_i' - q_i = 0$. Thus, $\{q'\} = \{q\}$ and, by (15), $n' = n$.

4. Superdeformed nuclei

Because of its application to superdeformed prolate nuclei, the particular case of

$$H = \hat{n}_1 + \hat{n}_2 + \tfrac{1}{2}\hat{n}_3 \tag{17}$$

is interesting to evaluate explicitly. Here the p_i are relatively prime and the maximal symmetry algebra is u(3). The new boson destruction operators are given in terms of the usual oscillator bosons by

$$\begin{aligned} A_1 &= a_1 \\ A_2 &= a_2 \\ A_3 &= a_3\hat{n}_3^{-1/2}a_3\hat{n}_3^{-1/2}[\hat{n}_3/2]^{1/2}. \end{aligned} \tag{18}$$

The highest weight vectors are just

$$\begin{aligned} &|n\rangle \otimes |0\rangle \otimes |0\rangle \\ &|n\rangle \otimes |0\rangle \otimes |1\rangle \end{aligned} \tag{19}$$

for $n = 0, 1, 2, \ldots$; the corresponding su(3) irreducible representations have identical weights $(\lambda, \mu) = (n, 0)$. In figure 1, the su(3) character of the anisotropic oscillator energy levels is indicated. In general, the decomposition of the anisotropic $(n, 0)$ irreducible representation as the vector span of isotropic states is

$$(n, 0) = \begin{cases} \operatorname*{span}_{0 \leq i \leq n} \{\text{isotropic } (n+i, 0)_{n+1-i}\} & E = n \\ \operatorname*{span}_{0 \leq i \leq n} \{\text{isotropic } (n+i+1, 0)_{n+1-i}\} & E = n + 1/2 \end{cases} \tag{20}$$

where the subscript on the isotropic su(3) representation indicates the number of vectors contributing to the anisotropic level $(n, 0)$. Hence, the number of spanning isotropic vectors at the level $E = n + \frac{1}{2}$ is

$$\sum_{i=0}^{n} (n + 1 - i) = \tfrac{1}{2}(n+1)(n+2) \tag{21}$$

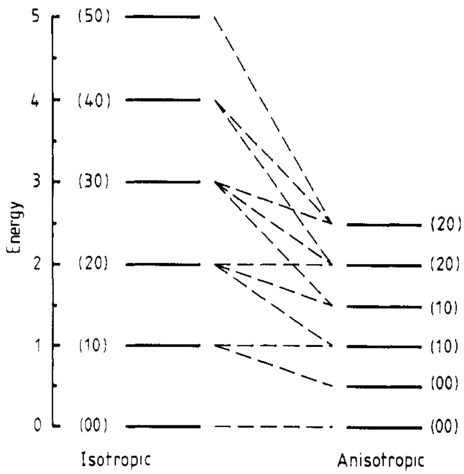

Figure 1. The energy spectrum of the anisotropic three-dimensional harmonic oscillator $H = n_1 + n_2 + n_3/2$ is drawn. The levels are labelled by their su(3) quantum numbers (λ, μ). Their parentage in terms of isotropic oscillator states is indicated by the broken lines.

which is the familiar dimension formula for the su(3) representation $(n, 0)$. Note that each anisotropic level is a mixture of even and odd parity states. If $E = n$, then the number of positive parity states is $([n/2]+1)^2$. If $E = n + \frac{1}{2}$, then the number of negative parity states is $([n/2]+1)^2$.

5. Dynamical algebra

The dynamical symmetry algebra for the anisotropic oscillator with commensurate frequencies whose reciprocals are relatively prime is the non-compact real symplectic Lie algebra $sp(m, \mathbb{R})$. In addition to the maximal compact subalgebra $u(m)$, the real symplectic algebra is generated by the raising and lowering operators

$$\begin{aligned} \mathscr{A}_{ij} &= \tfrac{1}{2} A_i(p_i)^\dagger A_j(p_j)^\dagger \\ \mathscr{B}_{ij} &\equiv \tfrac{1}{2} A_i(p_i) A_j(p_j). \end{aligned} \tag{22}$$

Every irreducible unitary representation of $sp(m, \mathbb{R})$ is infinite-dimensional. They are generated by the successive application of $\mathscr{A}_{ij}$ to the $u(m)$ representation spaces whose highest weight vectors are annihilated by the lowering operators B_{ij}, namely

$$\begin{aligned} &|n = 0, \{q\}\rangle \\ &|n = 1, \{q\}\rangle. \end{aligned} \tag{23}$$

Therefore, the entire infinite-dimensional space $\mathscr{L}^2(\mathbb{R}^m)$ is the direct sum of two irreducible unitary representations of $sp(m, \mathbb{R})$ generated from the $u(m)$ representations $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$, each of which occurs with multiplicity $\Pi_{i=1}^m p_i$.

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